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Renormalization group for bosons with internally generated cut-off

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Abstract. It is shown that the renormalization group approach can be extended to a Bose system without imposing an *ad hoc* upper momentum cut-off. The strategy adopted is to eliminate momenta between ∞ and a small but arbitrary momentum p_0 by performing a preliminary partial trace on the partition function. Recursion relations for the parameters of the resulting Hamiltonian are derived in a form which enables scaling functions for the susceptibility and the order parameter to be calculated. The functions exhibit a crossover from dilute gas critical behaviour to that of a fully interacting system. The results are in agreement with those derived by Weichman *et al* who relied upon a mapping of the Bose system onto a classical, two-component spin model. It is contended that a rigorous justification for the mapping emerges from the approach of this paper.

1. Introduction

An essential prerequisite for the application of the renormalization group (RG) theory of critical phenomena [1] to a system is that it should be characterized by an upper momentum cut-off p_c. For many physical systems such a cut-off exists naturally. For a lattice system, for instance, the dimension of the first Brillouin zone provides the upper limit. For a fluid system, such as an assembly of interacting bosons, on the other hand, no natural upper cut-off exists. In early attempts [2] to apply RG ideas to an assembly of bosons, a cut-off of the order of the thermal momentum λ_T^{-1} of the particles was imposed on the system in the expectation that momenta large compared to λ_T^{-1} would not be of much consequence for the critical behaviour of the system. Recently, experiments on the behaviour of superfluid helium in Vycor glass [3, 4] have renewed interest in the critical behaviour of a Bose system, especially as regards the problem of crossover from ideal-gas critical behaviour to that of a fully interacting system. The earlier RG treatments have been scrutinized, and in particular it has been argued [5] that the assumption $p_c \sim \lambda_T^{-1}$ is inappropriate in view of the fact that an important scaling parameter which enters the theory is

$$s = p_c^2 \lambda_T^2 / 4\pi \tag{1}$$

and this becomes independent of temperature for $p_c \sim \lambda_T^{-1}$. On the other hand, for a fixed p_c, s would go to infinity for $T \to 0$. It has been suggested that p_c should be taken of the order of $(1/a)$ where a is the mean distance between the particles. The above criticism of the magnitude of p_c does not constitute any serious deficiency of the theory on account of the fact that s appears as an *irrelevant* variable in the theory and consequently the original magnitude of s or p_c makes little difference to the theory.

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Nevertheless, what is perhaps objectionable is the imposition by hand of a cut-off on a system which in reality has no finite upper limit to momenta. The cut-off seems to be an artifice designed solely for the application of the RG theory to the system.

Is it possible to do away with the imposition of an *ad hoc* p_c? This question is answered in the affirmative in this paper by integration of momenta between infinity and an arbitrary low value p_0. Rescaling in the sense of RG is evidently not possible at this stage. The resulting effective Hamiltonian, however, can be subjected to complete RG transformations with p_0 playing the role of an upper fixed momentum. The parameters characterizing the effective Hamiltonian play an important role in the subsequent analysis because of the connections they provide to the parameters of the original Hamiltonian of the system.

Following the above procedure, recursion relations appropriate to the study of crossover behaviour are derived in section 3 in $(4-\varepsilon)$ dimensions. They show that the renormalized parameters which characterize the Hamiltonian after an RG transformation are functions of just one combination x of the scaling fields u and t which can be regarded as measures, respectively, of the strength of boson–boson interaction and deviation from criticality. As a result, scaling functions of various physical quantities are also functions of x only. This is demonstrated in section 4 by calculating the crossover scaling functions for the susceptibility and the order parameter.

The above results are in general agreement with those derived on the basis of a mapping of a Bose system onto a classical two-component spin model [5]. It should, however, be pointed out that the mapping has not been rigorously established in [5]. The arguments that have been advanced pertain to the normal phase only, leaving open the question of mapping for the condensed phase. As discussed in section 5, the RG approach of the present paper provides a more convincing justification of the mapping.

2. Generation of internal cut-off

The system under consideration is an assembly of interacting bosons contained in a d-dimensional box of volume $V=L^d$ and characterized by a temperature T and chemical potential μ. The reduced Hamiltonian of the assembly in units such that $\hbar=1$ is

$$H_0=\sum_k \beta\left[\frac{k^2}{m}-\mu\right]a_k^\dagger a_k+\frac{\beta u_0}{4V}\sum_{k_1\ldots k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_1+k_2-k_3} \tag{2}$$

where $(m/2)$ denotes the mass of a boson, $(u_0/2)$ the strength of boson–boson interaction, and β the inverse of the product of T and the Boltzmann constant k_B. The system is assumed to obey periodic boundary conditions which imply that each component k_i of single-particle momentum k has values $(2\pi n_i|L)$ with n_i ranging over the set of integers. In equation (2) the usual approximation of replacing the Fourier transform of the two-body potential by a constant u_0 has been made.

Our aim is to study the critical behaviour of the assembly by the RG method. The latter, as is well known, works only if the system possesses an upper momentum cut-off. Since a boson assembly has no natural cut-off, an appropriate way to introduce a cut-off is to perform a partial trace on the density matrix of the system so as to eliminate momenta between ∞ and some finite but arbitrary value p_0. Later on, it will be found convenient to choose p_0 small in comparison with the thermal momentum λ_T^{-1} given by

$$\lambda_T^{-1} = (m/4\pi\beta)^{1/2}. \tag{3}$$

The thermodynamic potential per unit volume of the assembly is given by

$$\Omega = -\frac{1}{\beta V}\ln Z \tag{4}$$

$$Z = \mathrm{Tr}\exp(-H_0). \tag{5}$$

In order to integrate out momenta between p_0 and ∞ we factorize the Hilbert space of the system as $h_0 \otimes h_1$ where h_0 denotes the subspace on which operators a_k with $|k| > p_0$ act, and h_1 the subspace on which a_k with $|k| < p_0$ act. Using the perturbation expansion of Z a partial trace over the subspace h_0 can be carried out. This procedure is similar to that explained in [2]. For a weak interaction it gives the result

$$Z = Z_0(\infty, \bar{p}_0) Z_1(\bar{p}_0) \tag{6}$$

where Z_0 denotes the contribution of connected graphs arising from the part of the Hamiltonian H_0 having momenta between p_0 and ∞, and Z_1 is given by

$$Z_1 = \mathop{\mathrm{Tr}}_{(h_1)} \exp[-H_1(r_1, s_1, v_1)] \tag{7}$$

$$H_1 = H_1^{(0)} + U_1 \tag{8}$$

$$H_1^{(0)} = s_1 \sum_{|k|<p_0} (k^2 p_c^{-2} + r_1) a_k^\dagger a_k \tag{9}$$

$$U_1 = \frac{s_1^2 p_0^{-d} v_1}{4V} \sum_{k_1,k_2,k_3} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_1+k_2-k_3}. \tag{10}$$

Each summation in (10) is restricted to the range $0 < |k| < p_0$. The parameters (s_1, r_1, v_1) are given by

$$s_1 = p_0^2 \lambda_T^2 / 4\pi \tag{11}$$

$$r_1 = s_1^{-1}\left[-\beta\mu + \frac{\beta u_0}{\lambda_T^d} A(s_1) + \mathrm{O}(\beta^2 u_0 \mu)\right] \tag{12}$$

$$v_1 = p_0^d s_1^{-2}[\beta u_0 + \mathrm{O}(\beta^2 u_0^2)] \tag{13}$$

$$A(s_1) = \frac{1}{\Gamma(d/2)} \int_{s_1}^{\infty} \frac{x^{d/2-1}}{\exp(x) - 1}\,\mathrm{d}x. \tag{14}$$

In what follows it will be convenient to choose p_0 small in comparison with λ_T^{-1} so as to ensure $s_1 \ll 1$. It is evident that a suitable choice of u_0 allows the parameter v_1 of the effective Hamiltonian to be treated as a small parameter.

3. Recursion relations

On the Hamiltonian H characterized by an upper cut-off p_0, we perform an RG transformation which eliminates moments in the range $p_0/\zeta < |k| < p_0$, $\zeta \gg 1$. This is achieved by expanding Z_1 as [2]

$$Z_1 = Z_{10} \mathop{\mathrm{Tr}}_{h(q)} \exp[-H_1^{(0)}(q)]\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^1 \dots \int_0^1 \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \langle P U_1(\tau_1) \dots U_1(\tau_n)\rangle\right]. \tag{15}$$

$$Z_{10} = \mathop{\mathrm{Tr}}_{h(p)} \exp[-H_1^{(0)}(p)]. \tag{16}$$

Here the subspace h_1 has been factorized as $h(p)\otimes h(q)$ where $h(p)$ denotes the subspace on which a_p with $|p|>p_0/\zeta$ act and $h(q)$ the subspace on which a_q with $|q|<p_0/\zeta$ act. $H_1^{(0)}(p)$ and $H_1^{(0)}(q)$ denote the two parts of $H_1^{(0)}$ which act on the subspaces $h(p)$ and $h(q)$ respectively. P denotes the time-ordering operator, $U_1(\tau)$ is given by

$$U_1(\tau)=\exp[\tau(H_1^{(0)}(q)+H_1^{(0)}(p))]U_1[-\tau(H_1^{(0)}(q)+H_1^{(0)}(p)] \tag{17}$$

and the angular bracket $\langle\cdot\rangle$ in equation (15) denotes themodynamic average calculated with the Hamiltonian $H_1^{(0)}(p)$.

The first-order term in (15) can be represented graphically as in figure 1. An external line represents a low momentum operator $a_q(\tau)$ or $a_q^\dagger(\tau)$ while an internal line represents a pairing of large momentum operators $a_p(\tau)$ and $a_p^\dagger(\tau)$. The contribution $C(a)$ of graph (a) in figure 1 is a constant given by

$$C(a)=\frac{Vs_1^2v_1}{2}p_0^dn'(\zeta) \tag{18}$$

$$n'(\zeta)=\int_{\zeta^{-1}}^{1}\mathrm{d}\boldsymbol{q}[\exp[s_1(q^2+r_1)]-1]^{-1}. \tag{19}$$

Throughout the paper $\mathrm{d}\boldsymbol{q}$ stands for $\mathrm{d}^dq/(2\pi)^d$.

The contribution of graph (b) of figure 1 is similar to $H_1^{(0)}(q)$. It adds to r_1 in $H_1^{(0)}$ a contribution $r_1^{(1)}$ which for small s_1, i.e. $p_0\ll\lambda_T^{-1}$, can be written as

$$r_1^{(1)}=A_1v_1[1-\zeta^{-2+\varepsilon}]-A_0r_1v_1\frac{(\zeta^{\varepsilon}-1)}{\varepsilon}+\mathrm{O}(r_1^2v_1) \tag{20}$$

where

$$A_1=A_0/(d-2) \tag{21}$$

$$A_0=2^{-d+1}\pi^{-d/2}/\Gamma_{(d/2)} \tag{22}$$

$$\varepsilon=4-d. \tag{23}$$

For the perturbation analysis that follows, ε has to be regarded small compared to 1. We note that to zeroth order in ε, A_0 equals $(1/8\pi^2)$.

The second-order term in (15) gives connected as well as disconnected graphs. The disconnected graphs can be shown to lead to exponentiation of the contribution of connected graphs [6]. The connected graphs are displayed in figure 2. Graphs $(2a)$ and $(2b)$ add constant terms of order v_1^2 to $C(a)$ given by (18). They are of no interest in what follows. Graphs $(2c)$ and $(2d)$ make contributions to the small-momentum vertex (c) of figure 1, while graphs $(2e)$ and $(2f)$ make contributions to r_1 in $H_1^{(0)}(q)$.

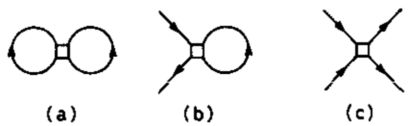

Figure 1. Graphical representation of the first-order term in equation (15). The external lines represent low momentum operators $(a_q, a_q^\dagger)$. The internal lines represent pairings of large momentum operators $(a_p, a_p^\dagger)$.

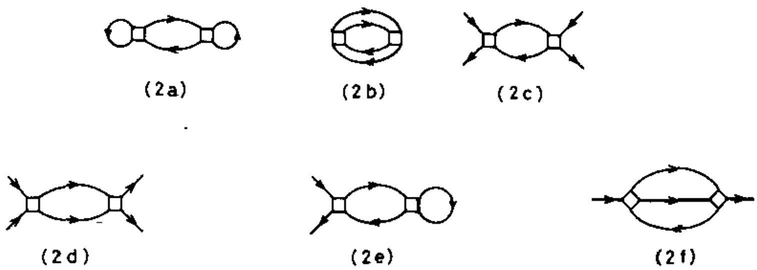

Figure 2. Connected graphs arising from the second-order term in equation (15).

The contributions of the graphs are calculated in the standard fashion [2]. Graphs (2c) and (2d) add to v_1 a term $v^{(2)}$ given by

$$v^{(2)} = -s_1 v_1^2 \int_{\zeta^{-1}}^{1} \mathrm{d}q \left[\frac{2 \exp E}{[\exp(E)-1]^2} + \frac{1}{4E} \frac{\exp(E)-1}{\exp(E)+1} \right] \tag{24}$$

$$E = s_1(q^2 + r_1). \tag{25}$$

For small s_1 this reduces to

$$v^{(2)} = -\tfrac{5}{2} A_0 v_1^2 [\ln \zeta + \mathrm{O}(s_1) + \mathrm{O}(r_1)]. \tag{26}$$

The effective small momentum vertex consequently acquires strength v_1' given by

$$v_1' = v_1 - \tfrac{5}{2} A_0 v_1^2 \ln \zeta. \tag{27}$$

The contribution of graph (2e) to r_1 in the limit of small s_1 is

$$C(2e) = -A_1 A_0 v_1^2 [1 - \zeta^{-2+\varepsilon}] \ln \zeta + \mathrm{O}(r_1 v_1^2) \tag{28}$$

while that of graph (2f) is

$$C(2f) = -(v_1^2/2) I(\zeta) + \mathrm{O}(r_1 v_1^2) \tag{29}$$

$$I(\zeta) = \iint \mathrm{d}\boldsymbol{q}_1 \, \mathrm{d}\boldsymbol{q}_2 [|\boldsymbol{q}_1 + \boldsymbol{q}_2|^2 q_1^2 q_2^2]^{-1} \tag{30}$$

where the integration is over the domain $\zeta^{-1} < (|\boldsymbol{q}_1|, |\boldsymbol{q}_2|) < 1$ subject to the restriction that $|\boldsymbol{q}_1 + \boldsymbol{q}_2|$ must also lie in the range $(\zeta^{-1}, 1)$. The latter restriction makes the calculation of I very difficult. As regards its dependence on ζ, however, it is not difficult to see [7] that to zeroth order in ε

$$I = \frac{A_0^2}{8} [a(1-\zeta^2) - b\zeta^{-2} \ln \zeta] \tag{31}$$

where a and b are pure numbers. Scaling arguments suggest [8] that $b = 12$. Combining (20), (28) and (29) and assuming v_1 and t to be of the same order of smallness, we find that the effective r_1 to second order is given by

$$r_1' = \left[r_1 + \frac{A_0}{2} v_1 + \mathrm{O}(\varepsilon v_1) + \mathrm{O}(v_1^2) \right] - \left[r_1 + \frac{A_0 v_1}{2} + \mathrm{O}(\varepsilon v_1) \right] A_0 v_1 \ln \zeta$$
$$- \frac{A_0}{2} \zeta^{-2+\varepsilon} [v_1 + \mathrm{O}(v_1^2)] - \tfrac{5}{2} A_0 v_1^2 \ln \zeta. \tag{32}$$

The partition function now takes the form

$$Z_1 = Z_{10}\exp(-C_0 + H_2) \tag{33}$$

where C_0 is a constant and

$$H_2 = s_1 \sum_q (q^2 p_c^{-2} + r_1')a_q^\dagger a_q + \frac{s_1^2 p_0^{-d}}{4V} \sum_{q_1 \dots q_4} v_1' a_{q_1}^\dagger a_{q_2}^\dagger a_{q_3} a_{q_4}. \tag{34}$$

To restore the range of q-momenta to $(0, p_0)$ they are scaled according to

$$q' = \zeta q. \tag{35}$$

The new q' are momenta of a particle in a box of volume

$$V_2 = \zeta^{-d} V = (L/\zeta)^d. \tag{36}$$

Defining new boson operators

$$b_{q'} = a_{(q'/\zeta)} \tag{37}$$

the Hamiltonian H_2 takes the same form as H_1. The parameters (s_2, r_2, v_2) of H_2 are related to (s_1, v_1, r_1) of H_1 by the relations

$$s_2 = \zeta^{-2} s_1 \tag{38}$$

$$r_2 = \zeta^2 r_1' \tag{39}$$

$$v_2 = \zeta^\varepsilon v_1'. \tag{40}$$

It is convenient to introduce a new variable

$$t_1 = r_1 + \frac{A_0}{2} v_1 \tag{41}$$

and let t_2 denote the same quantity after renormalization. Then (26), (32), (39) and (40) imply

$$t_2 = \zeta^2 t_1[1 - (v_1/8\pi^2)\ln\zeta] \tag{42}$$

$$v_2 = \zeta^\varepsilon \left[v_1 - \frac{5}{16\pi^2} v_1^2 \ln\zeta \right]. \tag{43}$$

In writing these relations A_0 has been replaced by $(1/8\pi^2)$.

The recursion relations (42) and (43) have a non-trivial fixed point

$$s^* = 0 \qquad t^* = 0 \qquad v^* = \frac{16\pi^2\varepsilon}{5}. \tag{44}$$

It is also evident that the fixed point is reached only if $v_1 = 0$ and $t_1 = 0$ because t_1, if not zero, increases continuously under RG transformations. The λ line is accordingly $t_1 = 0$.

To discuss crossover behaviour, one needs recursion relations which are valid not only when (t_1, v_1) are in the neighbourhood of the fixed point but also when they are far from it. The usual linearization of recursion relations near the fixed point is obviously not useful here.

We can write the recursion relations in a slightly different form by defining

$$u_i = v_i/v^* \qquad i = 1, 2. \tag{45}$$

Equations (42) and (43) can then be written as

$$u_2 = \zeta^\varepsilon u/(1+\zeta^\varepsilon u) \tag{46}$$

$$t_2 = \zeta^2 t/(1+\zeta^\varepsilon u)^{2/5} \tag{47}$$

where

$$u = \frac{u_1}{1-u_1} \tag{48}$$

$$t = t_1/(1-u_1)^{2/5}. \tag{49}$$

For small u_1, (u, t) are nearly the same as (u_1, t_1). Finally, choosing $\zeta^2 = |t|^{-1}$ we get

$$t_2 = \pm 1/(1+x)^{2/5} \tag{50}$$

$$u_2 = x/(1+x) \tag{51}$$

$$x = u/|t|^{\varepsilon/2}. \tag{52}$$

In (50), $\pm$ denotes the sign of t.

The recursion relations thus involve only a single combination x of the scaling fields u and t. As a consequence, various physical quantities will also be functions of x, as will be shown in the next section.

Near the critical line $t=0$, one can distinguish two limiting types of behaviour: the 'strong interaction' critical behaviour corresponding to $(t \to 0, u \neq 0)$ and the 'ideal gas' behaviour corresponding to $(|t| > 0, u \to 0)$. In the next section we shall calculate the equation of state of the system and show that it exhibits a crossover from ideal gas behaviour to non-classical critical behaviour.

4. Crossover functions

In this section we shall derive the equation of state for the system and exhibit its scaling property.

We start by adding an external-field term

$$-\frac{\beta h}{2} V^{1/2}(a_0 + a_0^\dagger) \tag{53}$$

to the dimensionless Hamiltonian H_0 of equation (2) where h denotes the field conjugate to the real part of the order parameter $a_0/\sqrt{V}$. Following Bogulubov [9] we replace $a_0/\sqrt{V}$ everywhere in the Hamiltonian by a real c-number M. The thermodynamic potential per unit volume is then given by

$$\Omega = \Omega'(T, \mu, M^2) - hM \tag{54}$$

$$\Omega' = -\frac{1}{\beta V} \ln \operatorname{Tr} \exp[-H_0(M)] \tag{55}$$

$H_0(M)$ denoting H_0 after the replacement $a_0/\sqrt{V} \to M$. Note that Ω' is a function of M^2 only by virtue of the invariance of H_0 under the gauge transformation $a_k \to a_k\, e^{i\pi}$, $M \to -M$. On minimizing Ω with respect to M, we get the equation of state

$$h = \frac{\partial \Omega'}{\partial M} = -\frac{1}{V}\left\langle \frac{\partial H_0}{\partial M} \right\rangle \tag{56}$$

and the condition of stability

$$\left[\frac{h}{M}+4M^2\frac{\partial^2\Omega'}{\partial M^2}\right]>0. \tag{57}$$

In (56) $\langle\cdot\rangle$ denotes thermodynamic average calculated with H_0.

The RG transformation does not involve the condensate operators $(a_0,\ a_0^\dagger)$. The results of section 2 consequently remain valid.

Equations (6), (33) and (56) imply

$$h_2=\frac{1}{V_2}\left\langle\frac{\partial}{\partial M_2}H_2(M_2)\right\rangle \tag{58}$$

where

$$h_2=\zeta^{d/2}h \tag{59}$$

$$M_2=\zeta^{d/2}M. \tag{60}$$

Comparison of (58) with (56) shows that the equation of state is formally invariant under the RG transformation.

The simplest way to evaluate the right-hand side of (56) is to use generalized Hartree–Fock factorization for the averages appearing in that expression. For small $(h|M)$ and small M, the result can be written as [10]

$$h_1/2M_1=r_n \tag{61}$$

where

$$h_1=\beta h(s_1p_0^d)^{-1/2} \tag{62}$$

$$M_1^2=s_1p_0^{-d}M^2 \tag{63}$$

$$r_n=t_1+v_1^2\left(I_1+\frac{I_2}{2}\right)+\tfrac{1}{2}v_1M_1^2 \tag{64}$$

$$r_s=\tfrac{1}{2}v_1M_1^2+\tfrac{1}{2}v_1I_2 \tag{65}$$

$$I_1=\frac{A_0}{2\varepsilon}[A(r_n)+A(r_n+2r_s)] \tag{66}$$

$$I_2=\frac{A_0}{2\varepsilon}[-A(r_n)+A(r_n+2r_s)] \tag{67}$$

$$A(r)=(r-r^{1-\varepsilon/2}) \tag{68}$$

and t_1 is the scaling variable defined in (41). The results assume that both r_n and (r_n+2r_s) are positive quantities; otherwise the spectrum of elementary excitations loses meaning. Note that by virtue of (57) r_n is greater than zero in the normal phase defined as $(h\to 0,\ M\to 0,\ h/M\neq 0)$. On performing an RG transformation, the equation of state stays invariant while $(r_n,\ r_s)$ scale as $\zeta^2(r_n,\ r_s)$ and M_1^2 scales as $\zeta^{d-2}M_1^2$. Choosing $\zeta^2=|t|^{-1}$ as before, the equation of state becomes

$$z_1=t_2+\tfrac{1}{2}u_2m^2+\frac{u_2}{10}[A(z_1)+3A(z_2)] \tag{69}$$

$$z_2=u_2m^2+z_1+\frac{u_2}{5}[-A(z_1)+A(z_2)] \tag{70}$$

where

$$z_1 = |t|^{-1} r_n \tag{71}$$

$$z_2 = |t|^{-1}(r_n + 2r_s) \tag{72}$$

$$m^2 = v^* M_1^2 |t|^{-1+\varepsilon/2}. \tag{73}$$

In the normal phase $z_2 \to z_1$ and (69) yields

$$z_1 = t_2(x) + \frac{2u_2}{5}[z_1 - z_1^{1-\varepsilon/2}]. \tag{74}$$

The sign of t_2 must be taken to be positive to ensure a positive z_1.

Equations (71) and (74) imply that the susceptibility r_n^{-1} has the form

$$r_n^{-1} = |t|^{-1} Y(x) \tag{75}$$

where the scaling function Y is given by

$$1 = \frac{Y}{(1+x)^{2/5}} - \frac{x\varepsilon}{5(1+x)} \ln Y. \tag{76}$$

The result holds for small ε, i.e. $\varepsilon \ln Y \ll 1$.

In the limit of a weakly interacting or dilute Bose gas ($u \to 0$, $t \neq 0$, $x \ll 1$) (75) implies $Y \simeq 1$ and therefore

$$r_n^{-1} = |t|^{-1}. \tag{77}$$

In the opposite limit of a strongly interacting system ($u \neq 0$, $t \to 0$, $x \gg 1$)

$$Y \simeq x^{2/5+2\varepsilon/5} \tag{78}$$

implying a susceptibility exponent

$$\gamma = 1 + \varepsilon/5. \tag{79}$$

The susceptibility thus shows a crossover from dilute-gas critical behaviour to that of a strongly interacting system. This result as well as equation (76) for Y are in agreement with those obtained by Weichman *et al* [5] by mapping a Bose system onto a classical S^4 spin model.

In the ordered phase ($h \to 0$, $m \neq 0$) equations (69) and (70) give

$$z_2 = \tfrac{2}{3}[m^2 u_2 - t_2(x)] \tag{80}$$

$$z_2 = m^2 u_2 + \frac{\varepsilon u_2}{10} z_2 \ln z_2. \tag{81}$$

Let $Y_2(x)$ denote the solution of these equations for m^2. The definition (73) then gives

$$M_1^2 v^* = |t|^{1-\varepsilon/2} Y_2(x). \tag{82}$$

Y_2 is thus the scaling function for the order parameter. Equations (80) and (81) are equivalent to the result derived in [5] for the crossover scaling function of the order parameter. The comparison is made more explicit by eliminating z_2 between (80) and (81) and defining

$$Q_1 = (m^2 x/2)^{5/3}. \tag{83}$$

The resulting equation for Q_1 to first order in ε is then

$$Q_1 = 1 + 2^{-\varepsilon/2} x Q_1^{\varepsilon/5}. \tag{84}$$

This is identical with the result derived in [5] if one writes Q_- there as

$$Q_- = (1 - u/u^*)Q_1 \tag{85}$$

and notes that our $M_1^2 v^*$ corresponds to $2m^2 u^*$ in [5].

In the limit of a strongly interacting system ($x \gg 1$) equations (80) and (81) have the solution

$$z_2 \simeq \tfrac{2}{3}(m^2 + x^{-2/5}) \tag{86}$$

$$m^2 \simeq 2^{1-3\varepsilon/10} x^{-2/5+3\varepsilon/25}. \tag{87}$$

The order parameter M in this case is proportional to $|t|^\beta$ with

$$\beta = \frac{1}{2} - \frac{3\varepsilon}{20}. \tag{88}$$

In the limit of a weakly interacting system ($x \ll 1$) we get

$$z_2 = \tfrac{2}{3}(m^2 x + 1) \tag{89}$$

$$m^2 = \frac{2}{x} - \frac{\varepsilon}{10} \ln 64 \tag{90}$$

which yields

$$M_1^2 v_1 \simeq 2|t|\left[1 - \frac{\varepsilon x}{10} \ln 64\right] \tag{91}$$

and hence $\beta = \frac{1}{2}$. Thus, like susceptibility, the order parameter also exhibits a crossover from dilute-gas critical behaviour to that of a strongly interacting system.

5. Discussion

The primary aim of this paper has been to show that the RG theory can be extended to a system of bosons without imposing an *ad hoc* upper momentum cut-off on the system. An appropriate mathematical procedure for this purpose is to carry out a partial trace which eliminates momenta between infinity and a low arbitrary momentum p_0. The resulting effective Hamiltonian can then be subjected to RG transformations in the usual manner.

The cut-off p_0 being simply an artifice, all physical results must be independent of p_0. This can be demonstrated by considering first the variable x which governs the crossover behaviour of various physical quantities. For small v_1/v^* equation (52) gives

$$x = \frac{v_1}{v^*} |t_1|^{-\varepsilon/2} \tag{92}$$

with v_1 and t_1 given by equations (13) and (41). For small s_1 the function $A(s_1)$ appearing in the expression (12) for r_1 has the form

$$A(s_1) = \zeta\left(\frac{d}{2}\right) - \frac{2}{(d-2)\Gamma(d/2)} s_1^{(d-2)/2}. \tag{93}$$

Consequently v_1 and t_1 take the form

$$v_1 = 16\pi^2 (p_0 \lambda_T)^{-\varepsilon} \beta u_0 / \lambda_T^d \tag{94}$$

$$t_1 = 4\pi (p_0 \lambda_T)^{-2} [-\beta\mu + \zeta(d/2) \beta u_0 / \lambda_T^d]. \tag{95}$$

It is now evident that x is independent of p_0. It is not difficult to check that z_1 and m^2 defined by (71) and (73) are also independent of p_0. Consequently, the equation of state given by (69) and (70) is independent of p_0.

The results derived in section 4 concerning crossover behaviour are in agreement with those derived by Weichman *et al* [5] who relied upon a mapping of the Bose system onto a two-component classical spin model. The alternative, fully quantum mechanical, analysis presented here is, however, important because the mapping referred to above has not been rigorously established. As a matter of fact, the equivalence established in [5] on the basis of a detailed matching of the perturbation expansions of the two systems holds for the normal phase only, and even there, one is obliged to 'trade off' non-zero Matsubara frequencies for a cut-off of order λ_T^{-1} for the momentum integrals. For the condensed phase of the system, no arguments in favour of the mapping have been adduced, presumably because perturbation expansions in the condensed phase of a Bose system tend to become more complex due to anomalous pairings.

We believe that it is the approach of this paper with an internally generated cut-off which provides the necessary backing to make the proof of the mapping onto a classical spin system rigorous. After several RG transformations $s_1 \to 0$. Consequently in the perturbation expansion of a Bose system, all Bose-like factors $\{\exp[s_1(E_k+r)]-1\}^{-1}$ arising from summations over the Matsubara frequencies $\omega_n = 2\pi n/\beta$ can be rigorously replaced by $\{s_1(E_k+r)\}^{-1}$. This means that in the limit $s_1 \to 0$, the quantum mechanical propagators corresponding to $\omega_n = 0$ are the only ones that survive. These propagators, however, are just the propagators appearing in the perturbation expansion of the classical spin system (cf [5]). The problem of divergent momentum integrals is no longer present because of the built-in cut-off p_0. The numerical factors arising from permutation of vertices etc in a diagram of a given order also become identical with those for the corresponding diagram in the two-component spin system [5].

The above arguments carry over to the perturbation expansions in the condensed phase on introducing the usual normal and anomalous self-energies Σ_{11} and Σ_{02} for the Bose system. As pointed out in earlier works [11], these self-energies are the analogues of longitudinal and transverse inverse susceptibilities in the perturbation expansion of the classical system.

To summarize, it is only after a preliminary partial integration which introduces an unambiguous cut-off p_0 and after a suitable number of RG transformations which make the parameter $s_1 \to 0$ that the perturbation expansion of a Bose system maps onto that of the classical two-component spin model.

The quantum mechanical RG has sometimes been criticized [5, 12] on the ground that the fixed point Hamiltonian H^* in this approach remains ill defined because $s^* = 0$ and V^* (for any finite starting value of the volume V) is also zero. This criticism, however, is only academic because as calculations in this paper and previous work [2, 11] demonstrate, nothing depends on the existence or otherwise of H^*. All that is necessary is that the parameter transformations (42) and (43) should have a well defined fixed point. It should be emphasized that a critical point is not a fixed point of the RG transformation but rather a point on the critical surface of the fixed point [13]. The quantum Hamiltonian is consequently well defined at the critical point. Non-existence of a well defined H^* may thus be considered as a characteristic feature of the quantum mechanical RG approach.

We also wish to state that the results derived in this paper are valid for all non-zero temperatures and consequently they hold in the low temperature limit characterized by holding T fixed and small, and treating μ as the independent variable. For fixed T the λ-line $t_1=0$ (cf equation (95)) becomes $\mu=\mu_c$ where

$$\beta\mu_c=\zeta\left(\frac{d}{2}\right)\left(\frac{m}{2\pi}\right)^d (k_B T)^{(d+2)/2}u_0. \tag{96}$$

The crossover scaling variable x can be written as

$$x=\frac{4\pi}{v^*}\left(\frac{m}{2\pi}\right)^d (k_B T)^{(d-2)/2}u_0/(\beta\mu-\beta\mu_c)^{\varepsilon/2} \tag{97}$$

the result being true for any T, large or small. Walasek [14] has used a field theoretic approach to derive a $T\to 0$ crossover scaling function for the equation of state. The scaling variable found in that reference is

$$y\propto T/h^{\varepsilon/2} \tag{98}$$

where h is proportional to $(\mu-\mu_c)$ (and not $(\mu-\mu_c)/\mu_c$ as has been pointed out in [5]). This result is in agreement with the general result (97).

In recent years attempts have been made to use the Feynman path integral method in statistical mechanics to study the critical behaviour of a system of interacting bosons. The starting point is an expression for the partition function as a functional integral over a 4D classical field $\hat{\psi}(\boldsymbol{r},\tau)$ [15]. This offers the advantage of treating the quantum and the corresponding classical system within the same framework. The results obtained so far [16] by this approach are equivalent to those obtained by the quantum mechanical RG but invoke an *ad hoc* upper cut-off momentum. The problem of crossover behaviour does not appear to have been attempted by the path integral approach.

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